Crossover and critical exponents for percolation in the semi-infinite plane

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Abstract. The percolation problem in the semi-infinite plane is discussed in terms of the two competing length scales $\xi(p)$, the correlation length, and d, the distance from the surface. We propose a crossover hypothesis for the percolation probability $P(d, \xi(p))$ and identify two limiting regimes. For $\xi(p) \ll d$ the critical behaviour is governed by the usual exponent β while for $\xi(p) \gg d$ a new critical exponent, β_s, is required. Using simple RSRG procedures we obtain a sequence of approximations to β_s/β and these are seen to show good agreement with a recent Monte Carlo simulation of the system. Finally we indicate how the techniques are applied to the semi-infinite three-dimensional case.

1. Introduction

The problem of site percolation may be treated using real space renormalisation group (RSRG) techniques (Reynolds *et al* 1977). Analogous methods exist for bond percolation (Young and Stinchcombe 1975). The original lattice, with site occupation probability p, is mapped onto one isomorphic to it, with an accompanying dilatation of length scale by a factor b. The renormalised probability of occupation is given by

$$p' = R_b(p). \tag{1.1}$$

For systems with dimensionality greater than one this recursion relation has a non-trivial unstable fixed point in $0 < p < 1$ which represents the critical density of sites at which an infinite cluster of connected sites appears. At this critical density one observes singular behaviour in such quantities as mean (finite) cluster size, root-mean-square distance between connected sites ('correlation length'), mean number of finite clusters per site, etc (Essam 1980). Each singularity is characterised by its respective *critical exponent*. Furthermore linearising the transformation $p' = R_b(p)$ about the critical point yields the *thermal* (rather than *magnetic*) eigenvalue λ, from which, for example, the 'correlation length' critical exponent ν may be derived in the usual way.

If, however, in two dimensions, the occupation probability for sites in the $z > 0$ upper half-plane is set to zero we are left with the problem of percolation in the semi-infinite plane. One may allow the occupation probability for percolation, or analogously, the coupling constant between sites in the Ising model, to be different at the surface to that in the bulk. Indeed, by doing so, workers have investigated the possibility of surface transitions in both semi-infinite percolation (De Bell 1979) and semi-infinite Ising systems (Dunfield and Noolandi 1980) in two and three dimensions.

In this paper we study site percolation in the semi-infinite plane, setting a uniform occupation probability p in $z \le 0$, and we apply Niemeyer–van Leeuwen type RSRG

757

techniques (Niemeyer and van Leeuwen 1974) to the system in the two limiting regimes $\xi(p) \gg d$ and $\xi(p) \ll d$. In particular we postulate that the *percolation probability P* obeys a scaling form and exhibits crossover behaviour between power law forms governed by the usual exponent β for sites deep in the bulk ($\xi(p) \ll d$) and a new exponent, β_s, as one approaches the surface ($\xi(p) \gg d$). We develop a sequence of approximations leading to improved estimates of the quantity β_s/β. Our results are consistent with this hypothesis and give good qualitative agreement with recent Monte Carlo simulations (Watson 1985, unpublished).

2. Scaling form and surface–bulk crossover

Consider a two-dimensional lattice of sites with site occupation probability p, 0 in $z \leq 0$ and $z > 0$ respectively. Define the bulk *correlation length* ξ as the root-mean-square distance between connected sites far from the surface, where the mean is taken only over sites in finite clusters. For a site near the surface, two competing characteristic length scales enter: its distance d from the surface is one, and $\xi(p)$ is the other. In such a case, we define the *percolation probability P* as the probability that a site A at $z = -d$ is occupied and that it is connected to an infinite number of other sites. In the terminology of phase transition theory this is the *order parameter* and it is non-zero only for $p > p_c$. Denote this probability as $P(d, \xi(p))$. This is the usual exact definition for $P(d, \xi(p))$; however, as we show below, it needs to be modified to make itself amenable to the scaling formalism.

From the above considerations we expect the percolation probability to satisfy the following scaling form in the neighbourhood of the critical point:

$$P(d, \xi(p)) = \begin{cases} C(p-p_c)^\beta F(\xi(p)/d), & p > p_c \\ 0, & p \leq p_c, \end{cases} \tag{2.1}$$

where $F(x)$ is a universal scaling function with the following limiting behaviour:

$$\begin{aligned} F(x) &\to 1, & \text{as } x \to 0, \\ F(x) &\to Bx^{-\alpha}, & \text{as } x \to \infty, \end{aligned} \tag{2.2}$$

and p_c satisfies $p_c = R_b(p_c)$.

Since $\xi(p) = D(p-p_c)^{-\nu}$ we observe that for $\xi(p) \gg d$

$$P(d, \xi(p)) = (BCD^{-\alpha}/d^{-\alpha})(p-p_c)^{\beta+\nu\alpha}. \tag{2.3}$$

This expression defines the *surface critical exponent*,

$$\beta_s = \beta + \nu\alpha. \tag{2.4}$$

If one imagines the semi-infinite system to have arisen from an infinite one from which a line of parallel bonds has been removed it is clear then that the scaling function $F(\xi(p)/d)$ may be interpreted as the following conditional probability:

$$F(\xi(p)/d) = \text{Prob(site at } z = -d \text{ is on infinite cluster in semi-infinite}$$
$$\text{plane|site is on infinite cluster in infinite plane).}$$

In particular then $F(x) \leq 1$, for all x, with equality only for $x \to 0$.

3. Real space renormalisation group approach for bulk and surface exponents

As mentioned above, the definition of the percolation probability must be expressed in a manner accessible to the RSRG procedure. One may regard $P(d, \xi(p))$ as the sum of probabilities of certain 'classes of configurations' ('events') of the sites on the lattice such that the site A is on the infinite cluster in the region $z \leq 0$. An approximate definition for $P(d, \xi(p))$ is obtained therefore by incorporating only a subset of these events into P. It follows that considering a large number of events will yield a more reliable $P(d, \xi(p))$ and therefore more reliable critical exponents will be derived from that definition. Such a generalised approach is given in § 4. In the present section we consider the crudest model with only one event as follows.

Definition 1. Site A is in the infinite cluster if:
 (i) it is occupied;
 (ii) it is in a cell which transforms to an occupied renormalised site;
 (iii) the renormalised site is in the infinite cluster of the renormalised lattice given that it is occupied.
This defines $P(d, \xi(p))$.

The infinite class of lattice configurations satisfying the above definition constitutes an event. Notice also that definition 1 is inductive.
 Denote $P_b(p) = P(d, \xi(p))$ and $P_s(p) = P(d, \xi(p))$ for $\xi(p) \ll d$ and $\xi(p) \gg d$ respectively.

3.1. Scaling within bulk, $\xi(p) \ll d$

Definition 1 was introduced by Thouless (1978) to deal with the bulk exponent β. We first discuss this case, i.e. the treatment of sites very deep in the bulk, $\xi(p) \ll d$. We use a blocking transformation and the rule R_0 to determine whether a renormalised site is occupied. The lattice of sites is partitioned into identical groups or cells of sites. A cell of sites in the original lattice is transformed into a site of the renormalised lattice. According to the rule R_0, a cell is transformed into an occupied renormalised site if and only if there is a cluster in the cell which spans the cell either horizontally or vertically.
 As shown by Thouless, blocking on the triangular lattice with length scale dilatation factor $b = \sqrt{3}$ (figure $1(a)$) yields

$$p' = p^3 + 3p^2(1-p). \tag{3.1}$$

This has a non-trivial fixed point at the exact value $p_c = \frac{1}{2}$ at which the percolation eigenvalue is $\lambda \equiv (dp'/dp)_{p_c} = \frac{3}{2}$. Definition 1 above then implies

$$P_b(p) = p[2p(1-p) + p^2]P_b(p')/p'. \tag{3.2}$$

Linearising this result near the percolation threshold, $\xi(p) \gg 1$, yields

$$P'/P \simeq \tfrac{4}{3}. \tag{3.3}$$

However, in the bulk $P_b(p) = C(p - p_c)^\beta$, $p > p_c$. From this we obtain $\lambda^\beta = \frac{4}{3}$. This then
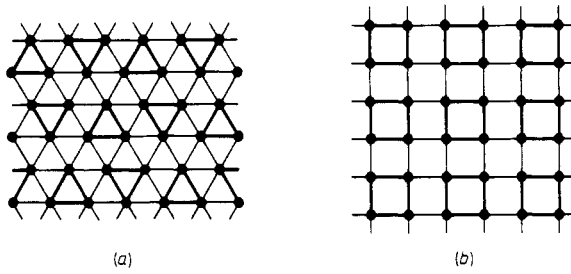
Figure 1. Blocks used for the real-space renormalisation of the site percolation problem in the bulk ($\xi(p) \ll d$) for (a) the triangular lattice with $b = \sqrt{3}$ and (b) the square lattice with $b = 2$.

gives for the bulk exponent β

$$\beta = (\log \tfrac{4}{3})/(\log \tfrac{3}{2}) \simeq 0.710. \tag{3.4}$$

The same procedure may be used on the square lattice when $b = 2$ (figure 1(b)). Here one obtains

$$P_b(p) = p[p^3 + 3p^2(1-p) + 2p(1-p)^2]P_b(p')/p' \tag{3.5}$$

and

$$p' = p^4 + 4p^3(1-p) + 4p^2(1-p)^2 \tag{3.6}$$

which yield $p_c = 0.382$, $\lambda = 1.53$ and so $\beta = 1.131$.

Comparing these values for β with the corresponding series result $\beta = 0.139 \pm 0.003$ (Blease *et al* 1978) highlights a twofold inadequacy in the above procedure. Firstly, considering only one event gives a severe underestimate for the numerical value of λ and hence overestimates β. This problem is a result of the crudeness of definition 1. However, it leads to a corresponding overestimate in β_s, the surface exponent, as we demonstrate below. This problem is to a certain extent eliminated if we are interested in the ratio of β and β_s. Indeed encouraging results are observed below for β_s/β. Secondly, the choice of such small cells in the blocking introduces an uncontrolled approximation due to cell interfacing problems. These interfacing problems become less significant if one uses larger blocking cells, i.e. larger b, as demonstrated by Reynolds *et al* (1980). One therefore expects definition 1 to yield improved results if one chooses larger cells in the blocking procedure.

We wish to determine whether a particular site is present in the infinite cluster. When dealing with larger cells an ambiguity arises in the positioning of the cell relative to our site of interest. An average over cell positions must be taken. It is convenient to introduce the *conditional RG transformation* defined as the probability that a renormalised site is occupied given that one of its constituent 'original' sites is occupied. Denote this by $C_{b,s}(p)$ for bulk and surface sites respectively. Notice that an average over original sites must be taken. Definition 1 then becomes

$$P_{b,s}(p)/p = C_{b,s}(p)P_{b,s}(p')/p'. \tag{3.7}$$

Now consider the $b = 3$ blocking on the square lattice for sites with $\xi(p)/d \ll 1$. Here we have

$$p' = p^9 + 9p^8(1-p) + 36p^7(1-p)^2 + 82p^6(1-p)^3 + 93p^5(1-p)^4$$
$$+ 44p^4(1-p)^5 + 6p^3(1-p)^6, \tag{3.8}$$

$$C_b(p) = p^8 + 8p^7(1-p) + 28p^6(1-p)^2 + (164/3)p^5(1-p)^3$$
$$+ (155/3)p^4(1-p)^4 + (176/9)p^3(1-p)^5 + 2p^2(1-p)^6. \qquad (3.9)$$

Using (3.7) above we obtain $\beta = 0.717$.

In a $b \times b$ cell on the square lattice there are $2^{b \times b}$ possible distinct configurations of original sites, a subset of which configurations are 'percolating'. To obtain the RG and conditional RG transformations one must determine the percolating configurations and weight each one appropriately. For $b = 4$ blocking, a FORTRAN program utilising the cluster multiple labelling algorithm of Hoshen and Kopelman (1976) was used to do this on a VAX 11/780 machine. Here an average over the 16 possible cell positions is taken for $C_b(s)$. The results are summarised in table 1.

Table 1. Results of successive approximations to β, β_s and β_s/β using RSRG blocking on various semi-infinite systems. (i) Definition 1: $P(d, \xi(p))/p = P(d/b, \xi(p'))C(p)/p'$. (ii) Definition 2: $P(d, \xi(p))/p = P(d/b, \xi(p'))C(p)/p' + P(d/b^2, \xi(p''))(1 - C(p))$ $D(p, p')/p''$. (iii) In 2D, Monte Carlo result for $\beta_s \simeq 0.41$; $\beta_s/\beta \simeq 2.95$ ($\beta \simeq 0.139$).

Lattice type	Defn of P	b	β	β_s	β/β_s
Triangular	1	$\sqrt{3}$	0.710	1.512	2.130
Triangular	2	$\sqrt{3}$	0.472	1.207	2.557
Triangular	1	2	0.924	2.129	2.304
Triangular	2	2	0.525	1.352	2.575
Square	1	2	1.131	1.630	1.441
Square	1	3	0.717	1.305	1.820
Square	1	4	0.568	1.197	2.106
Square (diagonal cut)	1	2	1.131	2.499	2.210
Square (diagonal cut)	1	3	0.717	1.800	2.511
Square (diagonal cut)	1	4	0.568	1.600	2.817
Square	2	2	0.709	1.271	1.793
Square (diagonal cut)	2	2	0.709	1.650	2.327
Simple cubic	1	2	2.099	2.431	1.157
Simple cubic (1:1:1 cut)	1	2	2.099	3.482	1.659

3.2. Scaling at surface ($\xi(p) \gg d$)

We now treat sites on the surface, representing the limit of the regime $\xi(p) \gg d$. The probabilistic interpretation above for the scaling function $F(\xi(p)/d) \leq 1$ implies $\beta_s \geq \beta$. The geometrical origin of this difference lies in the fact that if a site on the surface of the semi-infinite plane is to be connected to the infinite cluster, it is constrained to do so via sites that are *not* above it. This constraint is relieved for sites deep in the bulk. Thus $P_b > P_s$ and so $\beta_s \geq \beta$. We now proceed to evaluate β_s.

The surface constraint is incorporated naturally into the blocking procedure if one defines $C_s(p)$, the surface conditional RG transformation, as the probability that a surface cell is occupied given that one of its constituent surface sites is occupied. As when calculating $C_b(p)$, an average must be taken over possible positions of surface cell relative to the surface site; notice, though, that here certain surface cells contain sites that lie in $z > 0$ and that these sites are occupied with probability zero. This idea is illustrated in figure 2(a) for $b = \sqrt{3}$ surface blocking in the triangular lattice. In this case we obtain

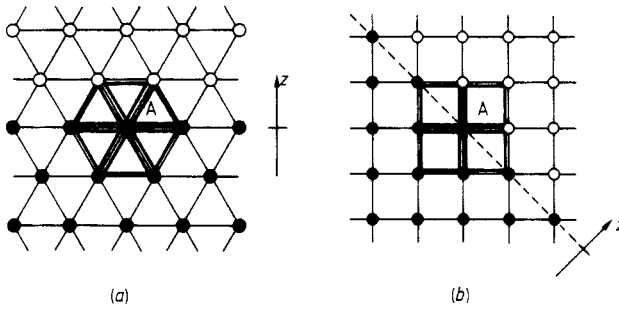$$C_s(p) = \tfrac{1}{6}[2p + 3(2p - p^2) + 0]. \qquad (3.10)$$

Figure 2. Surface blocking for RS renormalisation of the semi-infinite percolation problem for (a) the triangular lattice with $b = \sqrt{3}$ and (b) the square lattice (diagonal surface cut) with $b = 2$. The surface site A has Z (the coordination number) possible surface cells over which an average is to be taken. Sites in $z \leq 0$ and $z > 0$ are denoted by ●, ○ and are occupied with probability p, 0 respectively.

Using $P_s(p) = pC_s(p)P_s(p')/p'$ for the percolation probability in $\xi(p) \gg d$ it follows that $\beta_s \simeq 1.51$ which, using the corresponding result for β, gives $\beta_s/\beta \simeq 2.13$.

We perform the analogous procedure for a site on the surface of a square lattice. With $b = 2$ blocking we find

$$C_s(p) = \tfrac{1}{4}\{2p + 2[p^3 + 3p^2(1-p) + 2p(1-p)^2]\} \tag{3.11}$$

which yields $\beta_s \simeq 1.630$ and $\beta_s/\beta \simeq 1.441$.

Both the above results are poor when compared with that obtained in a Monte Carlo investigation of the same system $\beta_s/\beta \simeq 0.41/0.14 \simeq 2.9$. This is so for two reasons. Firstly, the spurious errors in both β_s and β arising from the small cell blocking and from the limitations of definition 1 for $P(d, \xi(p))$ compound each other. Secondly, we note that the renormalised surface site 'senses' the surface only as a result of some of its constituent sites having probability zero of being occupied. It is this property of the surface cells that embodies the essential physics of the semi-infinite system. However in, for example, the $b = 2$ surface blocking on the square lattice, two of the four distinct surface blocks do not then sense the surface at all and so contribute to keeping $C_s(p)$ artificially high and hence β_s low. This problem is inherent in small cell blocking and disappears when one uses larger cells. However, if one considers a semi-infinite square lattice with a surface along the diagonal (so that nearest neighbours on the surface are next-nearest neighbours of the lattice) we then expect (see figure 2(b)) the effect of the surface to be taken into account somewhat more faithfully. Universality arguments ensure that the physical value of β_s is left unchanged by this ploy.

We therefore treat $b = 2$ blocking on the square lattice with a diagonally cut surface. In the bulk β is unaffected while on the surface we have

$$C_s(p) = \tfrac{1}{4}\{2[p^2 + p(1-p)] + [p^3 + 3p^2(1-p) + 2p(1-p)^2] + 0\}. \tag{3.12}$$

Thus, using $P_s(p) = pC_s(p)P(p')/p'$ yields the result $\beta_s \simeq 2.50$ and $\beta_s/\beta \simeq 2.210$.

As mentioned above, it is possible to extract improved values of β_s/β if one uses larger cells. Scaling on the surface, when using larger cells, is analogous to that in the bulk with the additional requirement that sites present in the surface cells but in $z > 0$ are taken to have occupation probability zero. As an illustration we consider surface scaling in the $b = 3$ blocking of the square lattice with a diagonal cut. The surface

conditional RG transformation is in this case given by

$$
\begin{aligned}
C_s(p) = \tfrac{1}{9}\{ & [p^8 + 8p^7(1-p) + 28p^6(1-p)^2 + 55p^5(1-p)^3 \\
& + 51p^4(1-p)^4 + 18p^3(1-p)^5 + 2p^2(1-p)^6] + 2[p^7 + 7p^6(1-p) \\
& + 20p^5(1-p)^2 + 24p^4(1-p)^3 + 11p^3(1-p)^4 + p^2(1-p)^5] \\
& + [p^5 + 5p^4(1-p) + 4p^3(1-p)^2] + 2[p^5 + 5p^4(1-p) + 5p^3(1-p)^2] + 3(0) \}
\end{aligned}
$$
(3.13)

and we obtain $\beta_s \simeq 1.800$ and so $\beta_s/\beta \simeq 2.511$. The process is analogous for a straight cut. As anticipated, the $b = 4$ blocking on the square lattice gives results which are in even closer agreement with those obtained in the Monte Carlo simulation.

4. Extension of the renormalisation group approach: the 'two-event' definition

As one moves from small to larger cells it is apparent that β_s/β is approaching a limiting value. As explained above, the square lattice with a diagonal cut captures the surface properties well and so gives good results for β_s/β given the level of sophistication of the 'one-event' recursive definition of $P_s(p)$ and $P_b(p)$. The absolute values of β and β_s converge less quickly onto the expected limits. This convergence may be accelerated by including further events into the definition of P_b and P_s.

Definition 2. Site A is in the infinite cluster if:
either
    (i) it is occupied;
    (ii) it is in a cell which transforms to an occupied renormalised site;
    (iii) the renormalised site is in the infinite cluster of the renormalised lattice given that it is occupied;
or
    (i) it is occupied;
    (ii) it is in a cell which transforms to an unoccupied renormalised site;
    (iii) the (absent) renormalised site is part of a cell which transforms into a twice renormalised site that is occupied and present in the finite cluster of the twice renormalised lattice.

This redefines $P(d, \xi(p))$. Definition 2 is a two-event second-order recursive definition. The above two events are mutually exclusive and so one may add their respective probabilities. In principle one may extend this process to obtain more accurate definitions of $P(d, \xi(p))$.

In this case, we have, for bulk and surface sites respectively,

$$
P_{b,s}(p)/p = C_{b,s}(p)P_{b,s}(p')/p' + (1 - C_{b,s}(p))D_{b,s}(p, p')P_{b,s}(p'')/p'' \quad (4.1)
$$

where $p'' = R(p') = R^2(p)$ and $D(p, p')$ is the probability that a twice renormalised site is occupied given that one of its (once) renormalised sites is unoccupied.

We apply this result to the $b = \sqrt{3}$ blocking on the triangular lattice. In the bulk we have

$$
P_b(p)/p = (2p - p^2)P_b(p')/p' + [1 - (2p - p^2)]p'^2 P_b(p'')/p'' \quad (4.2)
$$

which on linearising yields $\beta \simeq 0.472$.

For sites at the surface we have

$$P_s(p)/p = \tfrac{1}{6}[2p + 3(2p - p^2) + 0]P_s(p')/p' + \{1 - \tfrac{1}{6}[2p + 3(2p - p^2) + 0]\}$$

$$\times \tfrac{1}{6}(2p'^2 + \tfrac{2}{3}p^2 p')P_s(p'')/p''. \tag{4.3}$$

From this it follows that $\beta_s \simeq 1.207$ and $\beta_s/\beta \simeq 2.557$.

Similar calculations are seen to give improved results when applied to $b = 2$ blocking on the triangular lattice and to blocking on the square lattice (see table 1).

5. Extension to the semi-infinite slab

The above ideas may in principle be applied to semi-infinite lattices in three dimensions. Unfortunately the computations, even using the relatively straightforward definition 1 for the percolation probability, are prohibitively involved for large cells. However, for completeness, we here present a $b = 2$ blocking calculation for the three-dimensional simple cubic lattice.

For sites in the bulk we obtain, using definition 1,

$$P_b(p)/p = C_b(p)P_b(p')/p', \tag{5.1}$$

$$p' = p^8 + 8p^7(1-p) + 28p^6(1-p)^2 + 56p^5(1-p)^3$$

$$+ 68p^4(1-p)^4 + 48p^3(1-p)^5 + 12p^2(1-p)^6, \tag{5.2}$$

$$C_b(p) = p^7 + 7p^6(1-p) + 21p^5(1-p)^2 + 35p^4(1-p)^3$$

$$+ 34p^3(1-p)^4 + 18p^2(1-p)^5 + 3p(1-p)^6. \tag{5.3}$$

While on the surface with a $1:1:1$ cut, for example, one obtains for the surface conditional probability $C_s(p)$

$$C_s(p) = \tfrac{1}{8}[C_b(p) + 3[p^3 + 2p^2(1-p) + p(1-p)^2]$$

$$+ 3[p^6 + 6p^5(1-p) + 15p^4(1-p)^2 + 19p^3(1-p)^3$$

$$+ 12p^2(1-p)^4 + p(1-p)^5]] \tag{5.4}$$

where $P_s(p)/p = C_s(p)P_s(p')/p'$.

The results of these and other calculations are summarised in table 1.

6. Discussion

Our approach to site percolation in semi-infinite systems is based on identifying $\xi(p)/d$ as a crossover variable in a scaling expression and thence determining the asymptotic ('bulk' and 'surface') behaviour of the system. In particular we have constructed RS scaling methods to yield the bulk and surface exponents, β and β_s respectively, for the percolation probability. The results, given in table 1, give convergence of the ratio β_s/β to a value in agreement with unpublished Monte Carlo data ($\beta_s/\beta \simeq 2.95$) obtained by B Watson in 1985 for $d = 2$. By elaborating the definition of the percolation probability we were able to obtain improved results for each value of cell size b.

In § 5 we demonstrated that the ideas are extended trivially to semi-infinite systems in three dimensions and results indicate that β_s/β is closer to 1 than in two dimensions. Although no numerical work is available to confirm this observation, it is consistent with the idea that the surface is less important in higher dimensions: the 'surface constraint' on sites at $z = 0$, that they must be connected to the infinite cluster via sites below them, becomes less stringent as the Euclidean dimensionality of the system is increased.

We now consider the relationship of β, β_s and the crossover exponent α to the structure of the infinite cluster in the semi-infinite system. If we enclose a portion of the infinite cluster within a hypercube of side L then the mass M of sites scales as

$$M \propto L^{d_f} \tag{6.1}$$

where d_f defines the fractal dimensionality (Mandelbrot 1982) of the infinite cluster. A simple scaling argument (e.g. Stauffer 1979) yields

$$d_f = d - \beta/\nu \tag{6.2}$$

where d is the Euclidean (or *embedding*) dimensionality.

It is useful to define $\tilde{\xi}(p)$, the RMS distance between sites on the surface connected via sites in $z \leq 0$, and the associated exponent $\tilde{\nu}_s$ given by

$$\tilde{\xi}(p) \sim (p - p_c)^{-\tilde{\nu}_s} \qquad \text{near } p_c. \tag{6.3}$$

If D_f is the fractal dimensionality of the set of sites on the infinite cluster at $z = 0$, an argument analogous to that for the bulk yields

$$D_f = (d - 1) - \beta_s/\tilde{\nu}_s. \tag{6.4}$$

In fact it may be shown that $\tilde{\nu}_s = \nu$ and so using the definition of the crossover exponent $\alpha = (\beta_s - \beta)/\nu$ we obtain

$$\alpha = (d_f - 1) - D_f. \tag{6.5}$$

Further, addition of codimensions (Mandelbrot 1982) yields the result that the fractal dimension of the set of sites in the intersection of the infinite cluster with a hyperplane is $d_f - 1$. We have therefore shown that α is the difference in dimension of the above set of sites and a subset of that set comprising those sites which are joined to an infinite number of sites on one particular side of the hyperplane.

Finally a study of the homogeneity properties of the appropriately defined generating functions indicates that the usual scaling relations between the percolation exponents describing the critical behaviour apply to surface as well as bulk exponents. Further work therefore needs to be done to determine the one remaining independent surface critical exponent.

References

Blease J, Essam J W and Place C M 1978 *J. Phys. C: Solid State Phys.* **11** 4009-16
De Bell K 1979 *J. Phys. C: Solid State Phys.* **12** L605-11
Dunfield L G and Noolandi J 1980 *Phys. Rev.* B **22** 2586-93
Essam J W 1980 *Rep. Prog. Phys.* **43** 835-912
Hoshen J and Kopelman R 1976 *Phys. Rev.* B **14** 3438-45
Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)
Niemeyer Th and van Leeuwen J M J 1974 *Physica* **71** 17-40
Reynolds P J, Klein W and Stanley H E 1977 *J. Phys. C: Solid State Phys.* **10** L167-72
Reynolds P J, Stanley H E and Klein W 1980 *Phys. Rev.* B **21** 1223-45
Stauffer D 1979 *Phys. Rep.* **54** 1
Thouless D J 1978 *Les Houches* eds R Balian, R Maynard and G Toulouse (Amsterdam: North-Holland) pp 1-62
Young A P and Stinchcombe R B 1975 *J. Phys. C: Solid State Phys.* **8** L535-40